**(1).** Decide whether the following statements are TRUE or FALSE. Justify your answers. State precisely the results that you use.

(a) The pull back of a trivial bundle is trivial.

(b) The tangent bundle of $\mathbb{S}^2$ splits as the Whitney sum of two line bundles.

(c) $\omega(\gamma^1) = 1 + a$.

(d) There exists a vector bundle $\eta$ such that $\gamma^1 \oplus \eta$ is trivial.

(e) The manifolds $\mathbb{RP}^2 \times \mathbb{RP}^3$ and $\mathbb{S}^5$ are cobordant.

**Solution**:
**(a)** True
**Justification:** Let $\varphi : E \to B \times V$ be a trivialization of a vector bundle $p : E \to B$. Let $f : X \to B$ be a continuous map. Then $\psi : f^*E \subseteq X \times E \xrightarrow{Id\times\varphi} X \times B \times V \to X \times V$ is a trivialization of $f^*E$ with inverse induced by the projection $g_1 : X \times V \to X$ and the map $g_2 : X \times V \xrightarrow{f\times Id} B \times V \xrightarrow{\varphi^{-1}} E$.

**(b)**False
**Justification:**

**Theorem 1** *(Classification of line bundles) The association $E \to \omega_1(E)$ gives a bijection between isomorphism classes of real line bundles on $B$ and $H^1(B; \mathbb{Z}_2)$.*

**Theorem 2** *(Hairy Ball Theorem) There is no nonvanishing continuous tangent vector field on even dimensional $n$-spheres.*

Suppose the tangent bundle of $\mathbb{S}^2$, $T\mathbb{S}^2 = L_1 \oplus L_2$, where $L_1$ and $L_2$ are line bundles over $\mathbb{S}^2$. By the classification of line bundles, $L_1$ and $L_2$ are trivial bundles. This implies that there is a nonvanishing continuous tangent vector field on $\mathbb{S}^2$. This is a contradiction by Hairy Ball Theorem, and therefore our assumption is wrong.

**(c)** True
**Justification:** Let $\gamma^1$ be the canonical line bundle over $\mathbb{RP}^\infty$. Then we have an obvious bundle map

$$\begin{array}{ccc} E(\gamma_1^1) & \longrightarrow & E(\gamma^1) \\ \downarrow{\scriptstyle \gamma_1^1} & & \downarrow{\scriptstyle \gamma^1} \\ \mathbb{S}^1 = \mathbb{RP}^1 & \xrightarrow{inclus} & \mathbb{RP}^\infty \end{array} \tag{1}$$

Therefore $i^*(\omega_1(\gamma^1)) = \omega_1(i^*(\gamma^1)) = \omega_1(\gamma_1^1) \neq 0$, by the axiom of the Stiefel-Whitney classes. This shows that $\omega_1(\gamma^1) = a$ is the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Hence $\omega(\gamma^1) = \omega_0(\gamma^1) + \omega_1(\gamma^1) = 1 + a$, since the bundle is one dimensional.

**(d)** False
**Justification:** Suppose there exists a bundle $\eta : E \to \mathbb{RP}^\infty$ such that $\gamma^1 \oplus \eta$ is trivial. Since $\omega(\gamma^1) = 1+a$,

where $a$ is the generator of $H^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$ and hence $\omega(E)$ must be $(1+a)^{-1} = 1 + a + a^2 + \cdots$. Since we are using $\mathbb{Z}_2$ coefficients. Thus $\omega_i(E) = a^i$, which is non-zero in $H^i(\mathbb{RP}^\infty;\mathbb{Z}_2)$for all $i$. However, this contradicts the fact that $\omega_i(E) = 0$ for $i > \dim E$.

**(e)** False
**Justification:**

**Theorem 3** *(Pontrjagin) If $B$ is a smooth compact $(n+1)$-dimensional manifold with boundary equal to $M$, then the Stiefel-Whitney numbers of $M$ are all zero.*

**Theorem 4** *Two smooth closed $n$-manifolds belong to the same cobordism class if and only if all of their corresponding Stiefel-Whitney numbers are equal.*

**Claim:** $\mathbb{RP}^2 \times \mathbb{RP}^3$ and $\mathbb{S}^5$ are not cobordant. Since $\omega_1(\mathbb{RP}^2 \times \mathbb{RP}^3) = \omega_1(\mathbb{RP}^2) \cup \omega_0(\mathbb{RP}^3) = 3a \cup 1 \neq 0$ in $H^1(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z}_2)$, where $a$ is the generator of $H^1(\mathbb{RP}^2;\mathbb{Z}_2)$ of $\mathbb{RP}^2 \times \mathbb{RP}^3$. So not all Stiefel-Whitney numbers are 0. But $\mathbb{S}^5 = \partial\mathbb{D}^6$. So, by Theorem 3 and Theorem 4, $\mathbb{RP}^2 \times \mathbb{RP}^3$ and $\mathbb{S}^5$ are not cobordant.

**(2).**
**(a).** Determine the least integer $k$ assuming that $\mathbb{RP}^9$ can be immersed in $\mathbb{R}^{9+k}$.

**Solution**: Now suppose that a manifold $M^n$ of dimension $n$ can be immersed in $\mathbb{R}^{n+k}$. Recall that a function $f : M^n \to \mathbb{R}^{n+k}$ is an immersion if at every point $x \in M^n$, $Df_x : T_xM \to \mathbb{R}^{n+k}$ is injective. This means that the tangent bundle $TM$ is an $n$-dimensional vector bundle, and the normal bundle $\nu$ is a $k$-dimensional vector bundle. From the Whitney Duality theorem, $\omega_i(\nu) = \overline{\omega_i}(M^n)$. So the dual Stiefel-Whitney classes $\overline{\omega_i}(M^n) = 0$ for $i > k$. For $M = \mathbb{RP}^9$,

$$\omega(\mathbb{RP}^9) = 1 + a^2 + a^8$$

and

$$\overline{\omega}(\mathbb{RP}^9) = 1 + a^2 + a^4 + a^6.$$

So if $\mathbb{RP}^9$ can be immersed in $\mathbb{R}^{9+k}$, then $k$ must be atleast 6.

**(b).** Show that the tangent bundle of $\mathbb{RP}^n$ has a sub bundle of rank 1 if and only if $n$ is odd.

**Solution**: Since $\omega(\mathbb{RP}^n) = (1+a)^{n+1}$. If $T(\mathbb{RP}^n) = \xi \oplus \eta$ for a line bundle $\xi$, then we must have $\omega(\xi)\omega(\eta) = \omega(\mathbb{RP}^n) = (1+a)^{n+1}$. $\omega(\xi)$ and $\omega(\eta)$ must be order 1 and $n-1$ respectively(by the axioms of Stiefel-Whitney classes) so it must be the case that $\omega(\xi) = 1 + a$ or 1. If $n$ is even, then in the first case, $\omega(\eta) = (1+a)^n$ has $\omega_n(\eta) = \binom{n}{n}a^n = a^n$, and in the later case $\omega_n(\eta) = \binom{n+1}{n}a^n = a^n$. This contradicts the fact that for an $(n-1)$-bundle such as $\eta$, $\omega_n(\eta) = 0$. Thus such a splitting cannot happen. If $n$ is odd, then $\omega_n(\mathbb{RP}^n) = 0$. This implies that $T(\mathbb{RP}^n)$ admits one global section by whose span defines a sub-bundle of dimension 1.
**(c).** Let $n = 2^k$. Find all the Stiefel-Whitney numbers of $\mathbb{RP}^n$.

**Solution**: $\omega(\mathbb{RP}^n) = (1+a)^{n+1} = (1+a)(1+a)^n$. Since we are working modulo 2 and $n = 2^k$. We have $(1+a)^n = 1 + a^n$. So $\omega(\mathbb{RP}^n) = (1+a)(1+a)^n = 1 + a + a^n + a^{n+1} = 1 + a + a^n$. Thus $\omega(\mathbb{RP}^n) = 1 + a + a^n$. This implies $\omega_0(\mathbb{RP}^n) = 1$, $\omega_1(\mathbb{RP}^n) = a$, $\omega_n(\mathbb{RP}^n) = a^n$ and $\omega_i(\mathbb{RP}^n) = 0 \;\; \forall\; i \neq 0, 1, n$. So the Stiefel-Whitney numbers $\langle \omega_n(\mathbb{RP}^n), [\mathbb{RP}^n] \rangle$ and $\langle \omega_1(\mathbb{RP}^n)^n, [\mathbb{RP}^n] \rangle$ are the only non-zero in $H^n(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$. All other Stiefel-Whitney numbers are zero.

**(d).** Let $k \geq 1$ be an integer and $f_k : S^1 \to S^1$ be the power map $z \mapsto z^k$. For which values of $k$ is the bundle ${f_k}^*(\gamma_1^1)$ trivial?

**Solution**: Let $\gamma_1^1$ be the canonical line bundle over $\mathbb{RP}^1 = \mathbb{S}^1$. Since $f_k : \mathbb{S}^1 \to \mathbb{S}^1$ has degree $k$, then $f_k^* : H^1(\mathbb{S}^1; \mathbb{Z}_2) \to H^1(\mathbb{S}^1; \mathbb{Z}_2)$ such that $f_k^*(x) = kx$. Suppose the pull back bundle $p : f_k^*(\gamma_1^1) \to \mathbb{S}^1$ is trivial.This implies that $0 = \omega_1(f_k^*(\gamma_1^1)) = f_k^*(\omega(\gamma_1^1)) = f_k^*(a) = ka$ where $a = \omega_1(\gamma_1^1)$ is the generator of $H^1(\mathbb{S}^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$. This shows that $k \equiv 0 \; mod \; 2$